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ADDENDUM

Proof of a General Relationship Used in the Stability Test of Linear Discrete Systems

In the stability criterion developed for the linear discrete systems, the following general relationship is used to reduce the stability constraints.[1]

$$\Delta_k = A_k^2 - B_k^2 = \frac{1}{2}[(A_{k+1} + B_{k+1})(A_{k-1} - B_{k-1}) + (A_{k+1} - B_{k+1})(A_{k-1} + B_{k-1})] \; , \quad k = 2, 3, \ldots, n-1 \qquad (1)$$

$$= A_{k-1}A_{k+1} - B_{k-1}B_{k+1}$$

where $A_k \pm B_k$ are given in the Appendix.

The proof of Eq. (1) is based on the combined use of the following three propositions. The reader should be familiar with Refs. 1 and 2 in order to follow the notations and the details of the proofs.

<u>Proposition 1:</u>

$\forall_n$, $\forall_k$, $\ni$ $2k-1 < n$, $A_k \pm B_k$ are prime (not factorable) polynomials in the ring of polynomials of n variables a_0, a_1, $a_2, \ldots,$ a_n.

<u>Proof:</u>

We readily notice that $A_1 + B_1$ is prime. Assume $A_{k-1} + B_{k-1}$ is prime, we show $A_k + B_k$ is prime too. The proof for $A_k - B_k$ is analogous to $A_k + B_k$.

Suppose, on the contrary, that $A_k + B_k$ is not prime. Let P and Q be polynomials into which $A_k + B_k$ factors,

$$A_k + B_k = PQ \qquad (2)$$

Expanding the determinant (see Appendix) for $A_k + B_k$ in cofactors of the last row we get:

$$PQ = A_k + B_k = a_0G + a_nG^* \qquad (3)$$

where G and G^* are the cofactors of a_0 and a_n.

By inspection, it can be seen that except for a relabeling of some indices, G with $a_n = 0$ and G^* with $a_0 = 0$ are of exactly the same form as $A_{k-1} + B_{k-1}$ (the restriction $2k - 2 < n$ is needed here and for later discussions we require $2k - 1 < n$). Therefore $G\big|_{a_n = 0}$ and $G^*\big|_{a_0 = 0}$ are prime.

If we put $a_n = 0$ in (3) and divide by P,

$$Q\Big|_{a_n = 0} = \frac{a_0 G\Big|_{a_n = 0}}{P\Big|_{a_n = 0}} \tag{4}$$

Also,

$$Q\Big|_{a_0 = 0} = \frac{a_n G^*\Big|_{a_0 = 0}}{P\Big|_{a_0 = 0}} \tag{5}$$

From (4) and (5) and the fact that Q is a polynomial and $G\big|_{a_n = 0}$, $G^*\big|_{a_0 = 0}$ are prime, it follows that P must be of the form:

$$P = (c_1 a_n + c_2 a_0 + c_3 + a_0 a_n P') \tag{6}$$

where P' is a polynomial and the c's are constants.

Similarly for Q we can write

$$Q = (c_4 a_n + c_5 a_0 + c_6 + a_0 a_n Q') \tag{7}$$

where Q' is a polynomial and the c's are constants.

Because $A_k + B_k$ is identically equal to PQ, all of the constants in (6) and (7) are necessarily zero. This implies $PQ \equiv 0$ if a_0 or $a_n = 0$ which is not true. Thus $A_k + B_k$ does not factor and hence a prime. By similar reasoning $A_k - B_k$ can be shown to be prime too.

Proposition 2:

$$\delta_i \pm a_{n-i}^{(i)} \text{ divides } \delta_{i+2} \mp a_{n-(i+2)}^{(i+2)}, \; i = 1,2,3,\ldots, n-1 \tag{8}$$

Proof: To prove Eq. (8), first we observe that the rules of calculating δ_{i+2} (see Table 1 of Ref. 2) from the previous two rows in the table and forming the sums $\delta_{i+2} \pm a_{n-(i+2)}^{(i+2)}$ are identical to those for calculating

δ_2 and $\delta_2 \pm a_{n-2}^{(2)}$ from the first two rows. Hence, the relationship, as far as divisibility is concerned, between $a_0 \pm a_n$ and $\delta_2 \mp a_{n-2}^{(2)}$ is exactly the same as that between $\delta_i \pm a_{n-i}^{(i)}$ and $\delta_{i+2} \mp a_{n-(i+2)}^{(i+2)}$. Therefore, to prove (8) it is sufficient to show that $a_0 \pm a_n$ divides $\delta_2 \mp a_{n-2}^{(2)}$. This can be shown as follows:

From Table (1) of Ref. (2),

$$\delta_2 = (a_0^2 - a_n^2)^2 - (a_0 a_{n-1} - a_1 a_n)^2 \tag{9}$$

and

$$a_{n-2}^{(2)} = (a_0^2 - a_n^2)(a_0 a_{n-2} - a_2 a_n) - (a_0 a_1 - a_{n-1} a_n)(a_0 a_{n-1} - a_1 a_n) \tag{10}$$

In forming $\delta_2 \pm a_{n-2}^{(2)}$, we obtain

$$\delta_2 - a_{n-2}^{(2)} = (a_0 + a_n)\phi_1 \tag{11}$$

and

$$\delta_2 + a_{n-2}^{(2)} = (a_0 - a_n)\phi_2 \tag{12}$$

where ϕ_1 and ϕ_2 are polynomials of the variables a_k's.

Therefore $a_0 \pm a_n$ divides $\delta_2 \pm a_{n-2}^{(2)}$ and this proves proposition 2.

<u>Proposition 3</u>:

$$\forall_n, \forall_k, \ni 2k - 1 < n,$$

$$\frac{\delta_k \pm a_{n-k}^{(k)}}{\delta_1^{k-2}\, \delta_2^{k-3}\, \delta_3^{k-4} \cdots \delta_{k-2}(A_{k-1} \mp B_{k-1})} = A_{k+1} \pm B_{k+1} \tag{13}$$

<u>Proof</u>: To prove the above relationship we introduce the following equation which can be obtained from Table 1 of Ref. 2:

$$\delta_{k+1} = (\delta_k + a_{n-k}^{(k)})(\delta_k - a_{n-k}^{(k)}),\ k = 2, 3, \ldots, n-1 \tag{14}$$

Furthermore, Marden[3] has proved the following identity in terms of the stability constants A_k's and B_k's.[1]

$$\Delta_{k+1} = (A_{k+1} + B_{k+1})(A_{k+1} - B_{k+1}) = \frac{\delta_{k+1}}{\delta_1^{k-1}\,\delta_2^{k-2}\,\delta_3^{k-3}\,\delta_{k-2}^{2}\,\delta_{k-1}} \qquad (15)^{+}$$

We assume that (13) is valid for all $k \leq m-1$; by induction we will show that (13) holds for $k = m$.

Using (15) we get for $k = m$

$$(A_{m+1} + B_{m+1})(A_{m+1} - B_{m+1}) = \frac{\delta_{m+1}}{\delta_1^{m-1}\,\delta_2^{m-2}\,\delta_3^{m-3}\cdots\delta_{m-2}^{2}\,\delta_{m-1}} \qquad (16)$$

Using (14) and (15) for δ_{m-1}, to obtain

$$(A_{m+1}+B_{m+1})(A_{m+1}-B_{m+1}) = \frac{(\delta_m + a_{n-m}^{(m)})(\delta_m - a_{n-m}^{(m)})}{(\delta_1^{m-1}\,\delta_2^{m-2}\,\delta_3^{m-3}\cdots\delta_{m-2}^{2})(\Delta_{m-1}\,\delta_1^{m-3}\,\delta_2^{m-4}\cdots\delta_{m-3})} \qquad (17)$$

The above equation can also be written as:

$$(A_{m+1} + B_{m+1})(A_{m+1} - B_{m+1}) = \underbrace{\left\{\frac{\delta_m + a_{n-m}^{(m)}}{\delta_1^{m-2}\,\delta_2^{m-3}\cdots\delta_{m-2}(A_{m-1}-B_{m-1})}\right\}}_{F_1}$$

$$\underbrace{\left\{\frac{\delta_m - a_{n-m}^{(m)}}{\delta_1^{m-2}\,\delta_2^{m-3}\cdots\delta_{m-2}(A_{m-1}+B_{m+1})}\right\}}_{F_2} \qquad (18)$$

+From this equation we can readily establish

$\delta_2 = \Delta_2$

$\delta_3 = \delta_1\delta_2 = \Delta_1\,\Delta_2$, since $\delta_1 = \Delta_1$)

$\delta_4 = \delta_1^2\,\delta_2\,\delta_3 = \Delta_1^2\,\Delta_2\,\Delta_1\,\Delta_2 = \Delta_1^3\,\Delta_2^2$

$\vdots$

$$\Delta_{k+1} = (A_{k+1} + B_{k+1})(A_{k+1} - B_{k+1}) = \frac{\delta_{k+1}}{\delta_1^{k-1}\,\delta_2^{k-2}\,\delta_3^{k-3}\,\delta_{k-2}^{2}\delta_{k-1}} \qquad (15)^{+}$$

We assume that (13) is valid for all $k \leq m-1$; by induction we will show that (13) holds for $k = m$.

Using (15) we get for $k = m$

$$(A_{m+1} + B_{m+1})(A_{m+1} - B_{m+1}) = \frac{\delta_{m+1}}{\delta_1^{m-1}\,\delta_2^{m-2}\,\delta_3^{m-3}\cdots\delta_{m-2}^{2}\delta_{m-1}} \qquad (16)$$

Using (14) and (15) for δ_{m-1}, to obtain

$$(A_{m+1}+B_{m+1})(A_{m+1}-B_{m+1}) = \frac{(\delta_m + a_{n-m}^{(m)})(\delta_m - a_{n-m}^{(m)})}{(\delta_1^{m-1}\delta_2^{m-2}\delta_3^{m-3}\cdots\delta_{m-2}^{2})(\Delta_{m-1}\delta_1^{m-3}\delta_2^{m-4}\cdots\delta_{m-3})}$$

(17)

The above equation can also be written as:

$$(A_{m+1} + B_{m+1})(A_{m+1} - B_{m+1}) = \underbrace{\left\{\frac{\delta_m + a_{n-m}^{(m)}}{\delta_1^{m-2}\delta_2^{m-3}\cdots\delta_{m-2}(A_{m-1}-B_{m-1})}\right\}}_{F_1}$$

$$\underbrace{\left\{\frac{\delta_m - a_{n-m}^{(m)}}{\delta_1^{m-2}\delta_2^{m-3}\cdots\delta_{m-2}(A_{m-1}+B_{m+1})}\right\}}_{F_2} \qquad (18)$$

[+] From this equation we can readily establish

$\delta_2 = \Delta_2$

$\delta_3 = \delta_1\delta_2 = \Delta_1\,\Delta_2$, since $\delta_1 = \Delta_1$)

$\delta_4 = \delta_1^2\,\delta_2\,\delta_3 = \Delta_1^2\,\Delta_2\,\Delta_1\,\Delta_2 = \Delta_1^3\,\Delta_2^2$

$\cdot \quad \cdot \quad \cdot$

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For simplicity we define F_1 and F_2 as indicated in the brackets of (18).

Next, we show that F_1 and F_2 are polynomials:

(a) $A_{m-1} \pm B_{m-1}$ and $[\delta_1^{m-2}\delta_2^{m-3}\ldots, \delta_{m-2}]$ are coprime (i.e., have no common factors).

By proposition 1, $A_{m-1} \pm B_{m-1}$ is prime. Using (15) for k = m-1,

$$[\delta_1^{m-2}\delta_2^{m-3}\ldots\delta_{m-2}]$$

can be written as a product of prime factors of the form $A_k \pm B_k$, $k<m-1$. Hence

$$[A_{m-1} \pm B_{m-1}] \quad \text{and} \quad [\delta_1^{m-2}\delta_2^{m-3}\ldots\delta_{m-2}]$$

are coprime.

(b) By assumption, (13) is valid for $k \leq m-1$ and in particular, $k = m-2$; therefore, by noting (13) we readily ascertain that $A_{m-1} \pm B_{m-1}$ is a divisor of

$$\delta_{m-2} \pm a_{n-(m-2)}^{(m-2)} .$$

Now if we apply proposition 2, it is clear that $A_{m-1} \pm B_{m-1}$ divides $\delta_m \mp a_{n-m}^{(m)}$ from (18).

(c) Since Δ_{m+1} is a polynomial, $\delta_m^2 - (a_{n-m})^2$ is divisible by

$$[\delta_1^{m-2}\delta_2^{m-3}\ldots\delta_{m-2}]^2 .$$

Furthermore,

$$\Delta_m = \frac{\delta_m}{\delta_1^{m-2}\delta_2^{m-3}\ldots\delta_{m-2}}$$

is a polynomial, it follows that $a_{n-m}^{(m)}$ and therefore $\delta_m \pm a_{n-m}^{(m)}$ are divisible by

$$\delta_1^{m-2}\delta_2^{m-3}\ldots\delta_{m-2} .$$

Items (b) and (c) in combination with (a) account for all the terms of the denominators dividing into the numerators in F_1 and F_2. Hence, F_1 and F_2 are polynomials.

From Proposition 1, $A_{m+1} \pm B_{m+1}$ are both prime. Since F_1F_2 in (18) is divisible by $A_{m+1} \pm B_{m+1}$, it follows that $A_{m+1}+B_{m+1}$ divides F_1 or F_2 and $A_{m+1} - B_{m+1}$ also divides F_1 or F_2. The degrees of $A_{m+1} \pm B_{m+1}$ and F_1 and F_2 are equal, so $A_{m+1} \pm B_{m+1}$ must be constant multiples of F_1 and F_2. Therefore, only these two possibilities may exist:

$$\text{I.} \qquad \left\{ \begin{aligned} A_{m+1} + B_{m+1} &= C_1F_1 \\ A_{m+1} - B_{m+1} &= D_1F_2 \end{aligned} \right\} \tag{19}$$

$$\text{II.} \qquad \left\{ \begin{aligned} A_{m+1} - B_{m+1} &= C_2F_1 \\ A_{m+1} + B_{m+1} &= D_2F_2 \end{aligned} \right\} \tag{20}$$

where C_1, D_1, C_2, and D_2 are constants.

From (18) we note

$$F_1(A_{m-1}-B_{m-1})+F_2(A_{m-1}+B_{m-1}) = \frac{2\delta_m}{\delta_1^{m-2}\delta_2^{m-3}\cdots\delta_{m-2}} \tag{21}$$

Using (15), we obtain

$$2\Delta_m = F_1(A_{m-1}-B_{m-1})+F_2(A_{m-1}+B_{m-1}) = 2(A_m-B_m)(A_m+B_m) \tag{22}$$

To show that for F_1 and F_2 only possibility (I) exists, we substitute in the above the second form, i.e., Eq. (20)

$$\begin{aligned} 2(A_m-B_m)(A_m+B_m) = {} & \frac{1}{C_2}(A_{m+1}-B_{m+1})(A_{m-1}-B_{m-1}) \\ & + \frac{1}{D_2}(A_{m+1}+B_{m+1})(A_{m-1}+B_{m-1}) \end{aligned} \tag{23}$$

To show the above does not hold for all m's, (1) we equate the coefficients of a_0^{2m} on both sides to obtain $2 = 1/C_2 + 1/D_2$; (2) we also equate the coefficients of a_n^{2m} on both sides. We notice that for m-<u>even</u>, the above (for finite C_2 and D_2) is not satisfied while possibility (I) is satisfied. Furthermore, by equating the coefficients of $a_0^2a_{n-1}^{2(m-1)}$ on both sides of the above equation in addition to (1) we find that for m-<u>odd</u> the above is also not satisfied while using (19), it is satisfied. Therefore (22) is satisfied only by using (19) for F_1 and F_2. Thus we finally obtain

$$A_m^2 - B_m^2 = \frac{1}{2}\left\{\frac{1}{C_1}(A_{m+1}+B_{m+1})(A_{m-1}-B_{m-1}) + \frac{1}{D_1}(A_{m+1}-B_{m+1})(A_{m-1}+B_{m-1})\right\} \tag{24}$$

We can show that $C_1=D_1=1$ for possibility (I) by equating the coefficients of a_m on both sides of Eq. (24).

$$0 = \pm\left(\frac{1}{C_1}a_n\Delta_{m-1} - \frac{1}{D_1}a_n\Delta_{m-1}\right)$$

or $1/C_1=1/D_1$ and from equating the coefficients of a_0^{2m} on both sides of Eq. (24), we get, $2=1/C_1+1/D_1$. Therefore, $C_1=D_1=1$.[+] This completes the proof of proposition 3.

If we substitute for m the variable k in (24),

$$\Delta_k = A_k^2 - B_k^2 = \frac{1}{2}\left\{(A_{k+1}+B_{k+1})(A_{k-1}-B_{k-1}) + (A_{k+1}-B_{k+1})(A_{k-1}+B_{k-1})\right\}$$

$$= A_{k-1}A_{k+1} - B_{k-1}B_{k+1} \tag{25}$$

which is (1) except for the restriction on k to be such that $2k-1 < n$. This restriction, which was only used in proving proposition 1 in fact, is not necessary. To see that (25) holds for all values of $k < n$, consider for the moment n to be fixed, say $n=n_0$. We have already shown (25) to be valid for all k such that $2k-1 < n_0$, and it remains to show its validity for all other $2k-1 \geq n_0$, $k < n_0$. For these values of k, consider (25) for $n=2n_0$. Since $k < n_0$, $2k-1 < n=2n_0$ so (25) is valid for these values of k and for $n=2n_0$. Put $a_{2n_0}=a_{n_0}$, $a_{2n_0-1}=a_{n_0-1}$, $a_{n_0+1}=a_1$. Then the resulting determinants for $k < n_0$ are identical to those for $n=n_0$. Hence, (25) is valid for all $k < n_0$, which proves the validity (1).

[+]By noting Eqs. (18) and (19), one can establish that $C_1D_1=1$. Using $C_1D_1=1$ in combination with $2=1/C_1+1/D_1$, we can readily establish that $C_1=D_1=1$.